MATH 20D Spring 2023 Lecture 15.

Introduction to the Laplace Transform

Announcements

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- Matlab Assignment 3 due this Friday.

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- Lecture for Friday May 12th and Monday May 15th will be recorded asynchronously and uploaded to Canvas. There will be **no** in person lecture on Friday May 12th and Monday May 15th.

Outline

Contents

Definition

Let $f\colon [0,\infty) \to \mathbb{R}$ *be a function. The* ***improper integral*** $\int_0^\infty f(t)dt$ *is defined as*

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Example

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(a) Calculate the improper integrals $\int_0^\infty \frac{dt}{1+t^2}$ and $\int_0^\infty \frac{dt}{1+t}$.

(b) Determine the values of $\alpha \in \mathbb{R}$ for which $I(\alpha) = \int_0^\infty e^{-\alpha t}dt$ converges.

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Using the integral definition above, calculate the Laplace transforms below

(a) $\mathscr{L}\{f_1\}$ where $f_1(t) = e^{at}$ and $a \in \mathbb{R}$ is constant.

(b) $\mathscr{L}\{f_2\}$ where $f_2(t) = \begin{cases} 2, & 0 \leqslant t < 5, \\ 0, & 5 \leqslant t < 10, \\ e^{4t}, & 10 \leqslant t < \infty. \end{cases}$

Proposition

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(a) Then $\mathscr{L}\{f_1 + f_2\}(s)$ exists for $s > s_0$ and

$$\mathscr{L}\{f_1 + f_2\}(s) = \mathscr{L}\{f_1\}(s) + \mathscr{L}\{f_2\}(s) \qquad \text{for all } s > s_0.$$

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(b) If c is constant then $\mathscr{L}\{f\}(s)$ exists for $s > s_0$ and

$$\mathscr{L}\{cf\}(s) = c\mathscr{L}\{f\}(s) \qquad \text{for all } s > s_0.$$

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Example

Let a be constant. Using the previous example, calculate the Laplace transform $\mathscr{L}\{f\}(s)$ where

$$f(t) = \begin{cases} 4 + 3e^{at}, & 0 \leqslant t < 5, \\ 3e^{at}, & 5 \leqslant t < 10, \\ 2e^{4t} + 3e^{at}, & 10 \leqslant t < \infty. \end{cases}$$

Let $f\colon [0,\infty) \to \mathbb{R}$ be a function. If $0 \leqslant a \leqslant b < \infty$ then the definite integral

$$\int_a^b f(t)dt$$

exists provided $f(t)$ is **piecewise continuous** over the interval $[a,b]$.

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Theorem

Suppose f is piecewise continuous. If there exist positive constants T and M such that

$$|f(t)| \leqslant Me^{\alpha t}, \qquad \text{for all } t \geqslant T$$

then $\mathscr{L}\{f\}(s) := \lim_{N\to\infty} \int_0^N e^{-st} f(t)dt$ converges for all $s > \alpha$.